

# ON THE HAUSDORFF AND PACKING MEASURES OF SLICES OF DYNAMICALLY DEFINED SETS

ARIEL RAPAPORT

**ABSTRACT.** Let  $1 \leq m < n$  be integers, and let  $K \subset \mathbb{R}^n$  be a self-similar set satisfying the strong separation condition, and with  $\dim K = s > m$ . We study the a.s. values of the  $s - m$ -dimensional Hausdorff and packing measures of  $K \cap V$ , where  $V$  is a typical  $n - m$ -dimensional affine subspace. For  $0 < \rho < \frac{1}{2}$  let  $C_\rho \subset [0, 1]$  be the attractor of the IFS  $\{f_{\rho,1}, f_{\rho,2}\}$ , where  $f_{\rho,1}(t) = \rho \cdot t$  and  $f_{\rho,2}(t) = \rho \cdot t + 1 - \rho$  for each  $t \in \mathbb{R}$ . We show that for certain numbers  $0 < a, b < \frac{1}{2}$ , for instance  $a = \frac{1}{4}$  and  $b = \frac{1}{3}$ , if  $K = C_a \times C_b$  then typically we have  $\mathcal{H}^{s-m}(K \cap V) = 0$ .

## 1. INTRODUCTION

Let  $1 \leq m < n$  be integers, and given  $0 \leq t \leq n$  let  $\mathcal{H}^t$  and  $\mathcal{P}^t$  be the  $t$ -dimensional Hausdorff and Packing measures respectively. Let  $s \in (m, n)$  be a real number, and let  $K \subset \mathbb{R}^n$  be compact with  $0 < \mathcal{H}^s(K) < \infty$ . Denote by  $\mu$  the restriction of  $\mathcal{H}^s$  to  $K$ , by  $G$  the set of all  $n - m$ -dimensional linear subspaces of  $\mathbb{R}^n$ , and by  $\xi_G$  the natural measure on  $G$ . It is well known that  $\dim_H(K \cap (x + V)) = s - m$  and  $\mathcal{H}^{s-m}(K \cap (x + V)) < \infty$ , for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$  (see Theorem 10.11 in [M1]). It is also known that if  $s = \dim_P K$  then  $\dim_P(K \cap (x + V)) \leq \max\{0, s - m\}$  for every  $V \in G$  and  $\mathcal{H}^m$ -a.e.  $x \in V^\perp$  (see Lemma 5 in [F1]), where  $\dim_P$  stands for the packing dimension. In this paper  $K$  will denote certain self-similar or self-affine sets, in which cases it will be shown that more can be said about the  $\mu \times \xi_G$ -typical values of  $\mathcal{H}^{s-m}(K \cap (x + V))$  and  $\mathcal{P}^{s-m}(K \cap (x + V))$ .

Assume first that  $K$  is a self-similar set which satisfies the strong separation condition (SSC). If  $m = 1$  and  $K$  is rotation-free, then from a result by Kempton (Theorem 6.1 in [K2]) it follows that  $\mathcal{H}^{s-m}(K \cap (x + V)) > 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V)$ , if and only if  $\frac{dP_{V^\perp} \mu}{d\mathcal{H}^m} \in L^\infty(dP_{V^\perp} \mu)$  for  $\xi_G$ -a.e.  $V$ , where  $P_{V^\perp}$  is the orthogonal projection onto  $V^\perp$ . In Theorem 1 below the case of a general  $1 \leq m < n$  and a general self-similar set  $K$ , satisfying the SSC, will be considered. A necessary and sufficient condition for  $\mathcal{H}^{s-m}(K \cap (x + V)) > 0$  to hold for  $\mu \times \xi_G$ -a.e.  $(x, V)$

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will be given. In Corollary 2 this condition is verified when  $m = 1$ ,  $s > 2$  and the rotation group of  $K$  is finite. Also given in Theorem 1, is a necessary and sufficient condition for  $\mathcal{H}^{s-m}(K \cap (x + V)) = 0$  to hold for  $\mu \times \xi_G$ -a.e.  $(x, V)$ .

Continuing to assume that  $K$  is a self-similar set with the SSC, it will be shown in Theorem 4 that  $\mathcal{P}^{s-m}(K \cap (x + V)) > 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V)$ . Also given in Theorem 4, is a sufficient condition for  $\mathcal{P}^{s-m}(K \cap (x + V)) = \infty$  to hold for  $\mu \times \xi_G$ -a.e.  $(x, V)$ . By using this condition, it is shown in Corollary 5 that this is in fact the case when  $m = 1$  and  $s > 2$ . This extends a result of Orponen (Theorem 1.1 in [O]), which deals with the case in which  $n = 2$ ,  $s > m = 1$  and  $K$  is rotation-free.

Lastly we consider the case in which  $n = 2$ ,  $m = 1$  and  $K$  is a certain self-affine set. For  $0 < \rho < \frac{1}{2}$  let  $C_\rho \subset [0, 1]$  be the attractor of the IFS  $\{f_{\rho,1}, f_{\rho,2}\}$ , where  $f_{\rho,1}(t) = \rho \cdot t$  and  $f_{\rho,2}(t) = \rho \cdot t + 1 - \rho$  for each  $t \in \mathbb{R}$ . It will be assumed that  $K = C_a \times C_b$ , where  $0 < a, b < \frac{1}{2}$  are such that  $a^{-1}$  and  $b^{-1}$  are Pisot numbers,  $\frac{\log b}{\log a}$  is irrational, and  $\dim_H(C_a) + \dim_H(C_b) > 1$ . Under these conditions it is shown in [NPS] that there exists a dense  $G_\delta$  set, of 1-dimensional linear subspaces  $V \subset \mathbb{R}^2$ , such that  $P_V \mu$  and  $\mathcal{H}^1$  are singular. By using this fact, it will be proven in Theorem 6 below that  $\mathcal{H}^{s-m}(K \cap (x + V)) = 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V)$ . This result demonstrates some kind of smallness of the slices  $K \cap (x + V)$ , hence it may be seen as related to a conjecture made by Furstenberg (Conjecture 5 in [F2]). In our setting this conjecture basically says that for  $\xi_G$ -a.e.  $V \in G$  we have  $\dim_H(K \cap (x + V)) \leq \max\{\dim_H K - 1, 0\}$  for each  $x \in \mathbb{R}^2$ , which demonstrates the smallness of the slices in another manner.

The rest of this article is organized as follows: In section 2 the results are stated. In section 3 the results regarding self-similar sets are proven. In section 4 we prove the aforementioned theorem regarding self-affine sets.

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## 2. STATEMENT OF THE RESULTS

**2.1. Slices of self-similar sets.** Let  $0 < m < n$  be integers, let  $G$  be the Grassmann manifold consisting of all  $n - m$ -dimensional linear subspaces of  $\mathbb{R}^n$ , let  $O(n)$  be the orthogonal group of  $\mathbb{R}^n$ , and let  $\xi_O$  be the Haar measure corresponding to  $O(n)$ . Fix  $U \in G$  and for each Borel set  $E \subset G$  define

$$(2.1) \quad \xi_G(E) = \xi_O\{g \in O(n) \mid gU \in E\},$$

then  $\xi_G$  is the unique rotation invariant Radon probability measure on  $G$ . For a linear subspace  $V$  of  $\mathbb{R}^n$  let  $P_V$  be the orthogonal projection onto  $V$ , let  $V^\perp$  be the orthogonal complement of  $V$ , and set  $V_x = x + V$  for each  $x \in \mathbb{R}^n$ .

Let  $\Lambda$  be a finite and nonempty set. Let  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$  be a self-similar IFS in  $\mathbb{R}^n$ , with attractor  $K \subset \mathbb{R}^n$  and with  $\dim_H K = s > m$ . For each  $\lambda \in \Lambda$  there exist  $0 < r_\lambda < 1$ ,  $h_\lambda \in O(n)$  and  $a_\lambda \in \mathbb{R}^n$ , such that  $\varphi_\lambda(x) = r_\lambda \cdot h_\lambda(x) + a_\lambda$  for each  $x \in \mathbb{R}^n$ . We assume that  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$  satisfies the strong separation condition. Let  $H$  be the smallest closed sub-group of  $O(n)$  which contains  $\{h_\lambda\}_{\lambda \in \Lambda}$ , and let  $\xi_H$  be the Haar measure corresponding to  $H$ . For each  $E \subset \mathbb{R}^n$  set  $\mu(E) = \frac{\mathcal{H}^s(K \cap E)}{\mathcal{H}^s(K)}$ , then  $\mu$  is a Radon probability measure which is supported on  $K$ .

For each  $0 \leq s < \infty$ ,  $\nu$  a Radon probability measure on  $\mathbb{R}^n$ , and  $x \in \mathbb{R}^n$  set

$$(2.2) \quad \Theta^{*s}(\nu, x) = \limsup_{\epsilon \downarrow 0} \frac{\nu(B(x, \epsilon))}{(2\epsilon)^s} \quad \text{and} \quad \Theta_*^s(\nu, x) = \liminf_{\epsilon \downarrow 0} \frac{\nu(B(x, \epsilon))}{(2\epsilon)^s},$$

where  $B(x, \epsilon)$  is the closed ball in  $\mathbb{R}^n$  with center  $x$  and radius  $\epsilon$ . It holds that  $\Theta^{*s}(\nu, \cdot)$  and  $\Theta_*^s(\nu, \cdot)$  are Borel functions (see remark 2.10 in [M1]). For  $V \in G$  define

$$F_V(x, h) = \Theta_*^m(P_{(hV)^\perp} \mu, P_{(hV)^\perp}(x)) \quad \text{for } (x, h) \in K \times H,$$

then  $F_V$  is a Borel function from  $K \times H$  to  $[0, \infty]$ . In what follows the collection  $\{F_V\}_{V \in G}$  will be of great importance for us.

Let  $\mathcal{V}$  be the set of all  $V \in G$  with

$$\xi_H(H \setminus \{h \in H : P_{(hV)^\perp} \mu \ll \mathcal{H}^m\}) = 0.$$

In Lemma 8 below it will be shown that  $\xi_G(G \setminus \mathcal{V}) = 0$ . First we state our results regarding the Hausdorff measure of typical slices of  $K$ .

**Theorem 1.** (i) Given  $V \in \mathcal{V}$ , if  $\|F_V\|_{L^\infty(\mu \times \xi_H)} < \infty$  then  $\mathcal{H}^{s-m}(K \cap (x + hV)) > 0$  for  $\mu \times \xi_H$ -a.e.  $(x, h) \in K \times H$ .

(ii) Given  $V \in \mathcal{V}$ , if  $\|F_V\|_{L^\infty(\mu \times \xi_H)} = \infty$  then  $\mathcal{H}^{s-m}(K \cap (x + hV)) = 0$  for  $\mu \times \xi_H$ -a.e.  $(x, h) \in K \times H$ .

(iii)  $\mathcal{H}^{s-m}(K \cap V_x) > 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$  if and only if  $\|F_V\|_{L^\infty(\mu \times \xi_H)} < \infty$  for  $\xi_G$ -a.e.  $V \in G$ .

(iv)  $\mathcal{H}^{s-m}(K \cap V_x) = 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$  if and only if  $\|F_V\|_{L^\infty(\mu \times \xi_H)} = \infty$  for  $\xi_G$ -a.e.  $V \in G$ .

From Theorem 1 we can derive the following corollaries.

**Corollary 2.** Assume  $m = 1$ ,  $s > 2$  and  $|H| < \infty$ , then  $\mathcal{H}^{s-m}(K \cap V_x) > 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$ .

**Corollary 3.** Assume that  $H = O(n)$  and

$$\mu \times \xi_G\{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0,$$

then there exists  $0 < M < \infty$  such that for each  $V \in G$  we have  $P_{V^\perp} \mu \ll \mathcal{H}^m$  with  $\left\| \frac{dP_{V^\perp} \mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} \leq M$ .

*Remark.* It is known that under the assumptions of Corollary 3 we have  $\dim(P_{V^\perp} \mu) = m$  for each  $V \in G$  (see Theorem 1.6 in [HS]). It is not known however if  $P_{V^\perp} \mu \ll \mathcal{H}^m$  for each  $V \in G$ , which is in fact a major open problem. Hence Corollary 3 implies that determining whether

$$\mu \times \xi_G \{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0$$

is probably quite hard.

Next we state our results regarding the packing measure of typical slices.

**Theorem 4.** (i)  $\mathcal{P}^{s-m}(K \cap V_x) > 0$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$ .  
(ii) Given  $V \in \mathcal{V}$ , if  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$  then  $\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty$  for  $\mu \times \xi_H$ -a.e.  $(x, h) \in K \times H$ .  
(iii) If  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$  for  $\xi_G$ -a.e.  $V \in G$ , then  $\mathcal{P}^{s-m}(K \cap V_x) = \infty$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$ .

From Theorem 4 the following corollary can be derived.

**Corollary 5.** Assume  $m = 1$  and  $s > 2$ , then  $\mathcal{P}^{s-m}(K \cap V_x) = \infty$  for  $\mu \times \xi_G$ -a.e.  $(x, V) \in K \times G$ .

*Remark.* In the proofs of Corollaries 2 and 5, we use the fact that if  $m = 1$  and  $s > 2$  then  $\frac{dP_{V^\perp} \mu}{d\mathcal{H}^m}$  is a continuous function for  $\xi_G$ -a.e.  $V \in G$  (see Lemma 3.2 in [FK] and the discussion before it). It is not known whether this is still true if  $m > 1$  or  $m < s \leq 2$ , hence we need the assumptions  $m = 1$  and  $s > 2$ .

**2.2. Slices of self-affine sets.** Assume  $n = 2$  and  $m = 1$ . Given  $0 < \rho < \frac{1}{2}$  define  $f_{\rho,1}, f_{\rho,2} : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_{\rho,1}(x) = \rho \cdot x \text{ and } f_{\rho,2}(x) = \rho \cdot x + 1 - \rho \text{ for each } x \in \mathbb{R},$$

let  $C_\rho \subset [0, 1]$  be the attractor of the IFS  $\{f_{\rho,1}, f_{\rho,2}\}$ , set  $d_\rho = \dim_H C_\rho$  (so that  $d_\rho = \frac{\log 2}{\log \rho^{-1}}$ ), and for each  $E \subset \mathbb{R}$  set  $\mu_\rho(E) = \frac{\mathcal{H}^{d_\rho}(C_\rho \cap E)}{\mathcal{H}^{d_\rho}(C_\rho)}$ .

**Theorem 6.** Let  $0 < a < b < \frac{1}{2}$  be such that  $\frac{1}{a}$  and  $\frac{1}{b}$  are Pisot numbers,  $\frac{\log b}{\log a}$  is irrational and  $d_a + d_b > 1$ , then  $\mathcal{H}^{d_a + d_b - 1}((C_a \times C_b) \cap V_{(x,y)}) = 0$  for  $\mu_a \times \mu_b \times \xi_G$ -a.e.  $(x, y, V) \in C_a \times C_b \times G$ .

*Remark.* Recall that every integer greater than 1 is a Pisot number, hence Theorem 6 applies for instance in the case  $a = \frac{1}{4}$  and  $b = \frac{1}{3}$ .

*Remark.* Note that  $0 < \mathcal{H}^{d_a + d_b}(C_a \times C_b) < \infty$ , see Lemma 18 below.

### 3. PROOF OF THE RESULTS ON SELF SIMILAR SETS

**3.1. Preliminaries.** The following notations will be used in the proofs of theorems 1 and 4. For each  $\lambda \in \Lambda$  set  $p_\lambda = r_\lambda^s$ . Then  $\mu$  is the unique self-similar probability measure corresponding to the IFS  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$  and the probability vector  $(p_\lambda)_{\lambda \in \Lambda}$ , i.e.  $\mu$  satisfies the relation  $\mu = \sum_{\lambda \in \Lambda} p_\lambda \cdot \mu \circ \varphi_\lambda^{-1}$ . Given a word  $\lambda_1 \cdot \dots \cdot \lambda_l = w \in \Lambda^*$  we write  $p_w = p_{\lambda_1} \cdot \dots \cdot p_{\lambda_l}$ ,  $r_w = r_{\lambda_1} \cdot \dots \cdot r_{\lambda_l}$ ,  $h_w = h_{\lambda_1} \cdot \dots \cdot h_{\lambda_l}$ ,  $\varphi_w = \varphi_{\lambda_1} \circ \dots \circ \varphi_{\lambda_l}$  and  $K_w = \varphi_w(K)$ . For each  $l \geq 1$  and  $x \in K$ , let  $w_l(x) \in \Lambda^l$  be the unique word of length  $l$  which satisfies  $x \in K_{w_l(x)}$ . Set also

$$(3.1) \quad \rho = \min\{d(\varphi_{\lambda_1}(K), \varphi_{\lambda_2}(K)) : \lambda_1, \lambda_2 \in \Lambda \text{ and } \lambda_1 \neq \lambda_2\},$$

then  $\rho > 0$  since  $\{\varphi_\lambda\}_{\lambda \in \Lambda}$  satisfies the strong separation condition. Given  $V_1, V_2 \in G$  set  $d_G(V_1, V_2) = \|P_{V_1} - P_{V_2}\|$  (where  $\|\cdot\|$  stands for operator norm), then  $d_G$  is a metric on  $G$ .

The following dynamical system will be used in the proofs of theorems 1 and 4. Set  $X = K \times H$  and for each  $(x, h) \in X$  let  $T(x, h) = (\varphi_{w_1(x)}^{-1}x, h_{w_1(x)}^{-1} \cdot h)$ . It is easy to check that the system  $(X, \mu \times \xi_H, T)$  is measure preserving, and from corollary 4.5 in [P] it follows that it is ergodic. Also, for  $k \geq 1$  and  $(x, h) \in X$  it is easy to verify that  $T^k(x, h) = (\varphi_{w_k(x)}^{-1}x, h_{w_k(x)}^{-1} \cdot h)$ .

Let  $\mathcal{R}$  be the Borel  $\sigma$ -algebra of  $\mathbb{R}^n$ . For each  $V \in G$  set  $\mathcal{R}_V = P_{V^\perp}^{-1}(\mathcal{R})$ , and let  $\{\mu_{V,x}\}_{x \in \mathbb{R}^n}$  be the disintegration of  $\mu$  with respect to  $\mathcal{R}_V$  (see section 3 of [FH]). For  $\mu$ -a.e.  $x \in \mathbb{R}^n$  the probability measure  $\mu_{V,x}$  is defined and supported on  $K \cap V_x$ . Also, for each  $f \in L^1(\mu)$  the map that takes  $x \in \mathbb{R}^n$  to  $\int f d\mu_{V,x}$  is  $\mathcal{R}_V$ -measurable, the formula

$$\int f d\mu = \int \int f(y) d\mu_{V,x}(y) d\mu(x)$$

is satisfied, and for  $\mu$ -a.e.  $x \in V^\perp$  we have

$$\int f d\mu_{V,x} = \lim_{\epsilon \downarrow 0} \frac{1}{P_{V^\perp} \mu(B(x, \epsilon))} \cdot \int_{P_{V^\perp}^{-1}(B(x, \epsilon))} f d\mu.$$

For more details on the measures  $\{\mu_{V,x}\}_{x \in \mathbb{R}^n}$  see section 3 of [FH] and the references therein.

**3.2. Auxiliary lemmas.** We shall now prove some lemmas that will be needed later on. The following lemma will be used with  $\xi_H$  in place of  $\eta$ , when  $\xi_H$  is considered as a measure on  $O(n)$  (which is supported on  $H$ ).

**Lemma 7.** *Let  $Q$  be a compact metric group, and let  $\nu$  be its normalized Haar measure. Let  $\eta$  be a Borel probability measure on  $Q$ , then for each Borel set  $E \subset Q$*

$$\nu(E) = \int_Q \eta(E \cdot q^{-1}) d\nu(q).$$

*Proof:* For each Borel set  $E \subset Q$  define  $\zeta(E) = \int_Q \eta(E \cdot q^{-1}) d\nu(q)$ . Since  $\nu$  is invariant it follows that for each  $g \in Q$

$$\zeta(Eg) = \int_Q \eta(E \cdot g \cdot q^{-1}) d\nu(q) = \int_Q \eta(E \cdot g \cdot (q \cdot g)^{-1}) d\nu(q) = \zeta(E).$$

This shows that  $\zeta$  is a right-invariant Borel Probability measure on  $Q$ , hence  $\nu = \zeta$  by the uniqueness of the Haar measure, and the lemma follows.  $\square$

**Lemma 8.** *Let  $\mathcal{V}$  be the set of all  $V \in G$  with*

$$\xi_H(H \setminus \{h \in H : P_{(hV)^\perp} \mu \ll \mathcal{H}^m\}) = 0,$$

*then  $\xi_G(G \setminus \mathcal{V}) = 0$ .*

*Proof of Lemma 8:* Set  $L = G \setminus \{V \in G : P_{V^\perp} \mu \ll \mathcal{H}^m\}$ . Since  $s > m$  it follows that  $I_m(\mu) < \infty$  (where  $I_m(\mu)$  is the  $m$ -energy of  $\mu$ ), hence from theorem 9.7 and equality (3.10) in [M1] we get that  $\xi_G(L) = 0$ . Let  $U \in G$  be as in (2.1) and set  $L' = \{g \in O(n) : gU \in L\}$ , then  $\xi_O(L') = \xi_G(L) = 0$ . Let  $B \subset O(n)$  be a Borel set with  $L' \subset B$  and  $\xi_O(B) = 0$ , then from Lemma 7 it follows that

$$0 = \xi_O(B) = \int \xi_H(B \cdot g^{-1}) d\xi_O(g).$$

From this we get that for  $\xi_O$ -a.e.  $g \in O(n)$

$$\begin{aligned} 0 = \xi_H(B \cdot g^{-1}) &\geq \xi_H(L' \cdot g^{-1}) = \xi_H\{h \in H : hg \in L'\} = \\ &= \xi_H(H \setminus \{h \in H : P_{(hgU)^\perp} \mu \ll \mathcal{H}^m\}), \end{aligned}$$

and so

$$\xi_H(H \setminus \{h \in H : P_{(hV)^\perp} \mu \ll \mathcal{H}^m\}) = 0 \text{ for } \xi_G\text{-a.e. } V \in G,$$

which proves the lemma.  $\square$

**Lemma 9.** *Let  $\mathcal{Z}$  be the set of all  $(x, V) \in K \times G$  such that  $\mu_{V,x}$  is defined and*

$$\mu_{V,x}(K_w) = \lim_{\epsilon \downarrow 0} \frac{\mu(K_w \cap P_{V^\perp}^{-1}(B(P_{V^\perp}x, \epsilon)))}{P_{V^\perp} \mu(B(P_{V^\perp}x, \epsilon))} \text{ for each } w \in \Lambda^*,$$

*then for each  $V \in G$  we have*

$$\mu \times \xi_H\{(x, h) \in X : (x, hV) \notin \mathcal{Z}\} = 0.$$

*Proof:* Fix  $V \in G$ . It holds that  $\mathcal{Z}$  is a Borel set, see section 3 of [M2] for a related argument. It follows that the set

$$\mathcal{Z}_V = \{(x, h) \in X : (x, hV) \in \mathcal{Z}\}$$

is also a Borel set. From the properties stated in section 3.1 we get that

$$\mu\{x \in K : (x, h) \notin \mathcal{Z}_V\} = 0 \text{ for each } h \in H,$$

and so  $\mu \times \xi_H(X \setminus Z_V) = 0$  by Fubini's theorem. This proves the lemma.  $\square$

**Lemma 10.** *Given a compact set  $\tilde{K} \subset \mathbb{R}^n$  and  $0 < t \leq n$ , the map that takes  $(x, V) \in \tilde{K} \times G$  to  $\mathcal{H}^t(\tilde{K} \cap V_x)$  is Borel measurable.*

*Proof:* For  $\delta > 0$  let  $\mathcal{H}_\delta^t$  be as defined in section 4.3 of [M1]. Let  $(x, V) \in \tilde{K} \times G$ ,  $\epsilon > 0$  and  $\{(x_k, V^k)\}_{k=1}^\infty \subset \tilde{K} \times G$ , be such that  $(x_k, V^k) \xrightarrow{k} (x, V)$ . Let  $W_1, W_2, \dots \subset \mathbb{R}^n$  be open sets with  $\tilde{K} \cap V_x \subset \bigcup_{j=1}^\infty W_j$ ,

$$\sum_{j=1}^\infty (\text{diam}(W_j))^t \leq \mathcal{H}_\delta^t(\tilde{K} \cap V_x) + \epsilon$$

and  $\text{diam}(W_j) \leq \delta$  for each  $j \geq 1$ . Since  $\tilde{K}$  is compact and since  $(x_k, V^k) \xrightarrow{k} (x, V)$ , it follows that  $\tilde{K} \cap V_{x_k}^k \subset \bigcup_{j=1}^\infty W_j$  for each  $k \geq 1$  which is large enough, and so for each such  $k$

$$\mathcal{H}_\delta^t(\tilde{K} \cap V_{x_k}^k) \leq \sum_{j=1}^\infty (\text{diam}(W_j))^t < \mathcal{H}_\delta^t(\tilde{K} \cap V_x) + \epsilon.$$

It follows that the function that maps  $(x, V)$  to  $\mathcal{H}_\delta^t(\tilde{K} \cap V_x)$  is upper semi-continuous, and so Borel measurable. Now since  $\mathcal{H}^s = \lim_{k \rightarrow \infty} \mathcal{H}_{1/k}^s$  the lemma follows.  $\square$

**Lemma 11.** *Given  $0 < t \leq n$  and a Radon probability measure  $\nu$  on  $K \times G$ , the map that takes  $(x, V) \in K \times G$  to  $\mathcal{P}^t(K \cap V_x)$  is  $\nu$ -measurable (i.e. this map is universally measurable).*

*Proof:* Let  $a \geq 0$  and set  $E = \{(x, V) \in K \times G : \mathcal{P}^t(K \cap V_x) < a\}$ , then in order to prove the lemma it suffice to show that  $E$  is  $\nu$ -measurable.

Set  $Y = \{C \subset K : C \text{ is compact}\}$ , endow  $Y$  with the Hausdorff metric, and let  $\mathcal{G}$  be the  $\sigma$ -algebra of  $Y$  which is generated by its analytic subsets. Set

$$\mathcal{E} = \{C \in Y : \mathcal{P}^t(C) < a\},$$

then from Theorem 4.2 in [MM] it follows that  $\mathcal{E} \in \mathcal{G}$ , and so from Theorem 21.10 in [K1] we get that  $\mathcal{E}$  is universally measurable.

For each  $(x, V) \in K \times G$  set  $\psi(x, V) = K \cap V_x$ , it will now be shown that  $\psi : K \times G \rightarrow Y$  is a Borel function. For each  $y \in K$  the function that maps  $(x, V) \in K \times G$  to  $d(K \cap V_x, y)$  is lower semi-continuous, and hence a Borel function. For each  $l \geq 1$  let  $S_l \subset K$  be finite and  $l^{-1}$ -spanning, and set  $\psi_l(x, V) = \{y \in S_l : d(K \cap V_x, y) \leq l^{-1}\}$  for each  $(x, V) \in K \times G$ . It holds that  $\psi_l : K \times G \rightarrow Y$  is a Borel function and  $\psi_l \xrightarrow{l \rightarrow \infty} \psi$  pointwise, hence  $\psi$  is a Borel function. Note also that  $E = \psi^{-1}(\mathcal{E})$ .

Since  $\mathcal{E}$  is universally measurable it is  $\nu \circ \psi^{-1}$ -measurable, and so there exist  $\mathcal{A}$  and  $\mathcal{C}$ , Borel subsets of  $Y$ , such that  $\mathcal{A} \subset \mathcal{E} \subset \mathcal{C}$  and  $\nu \circ \psi^{-1}(\mathcal{C} \setminus \mathcal{A}) = 0$ . It holds that  $\psi^{-1}(\mathcal{A})$  and  $\psi^{-1}(\mathcal{C})$  are Borel subsets of  $K \times G$ ,  $\psi^{-1}(\mathcal{A}) \subset E \subset \psi^{-1}(\mathcal{C})$  and

$\nu(\psi^{-1}(\mathcal{C}) \setminus \psi^{-1}(\mathcal{A})) = 0$ . This shows that  $E$  is  $\nu$ -measurable, and the lemma is proved.  $\square$

**Lemma 12.** *For  $(x, h, V) \in K \times H \times G$  set  $\psi(x, h, V) = (x, hV)$  and let  $B \in K \times G$  be universally measurable. Assume that for  $\xi_G$ -a.e.  $V \in G$  it holds for  $\xi_H$ -a.e.  $h \in H$  that*

$$\mu\{x \in K : \psi(x, h, V) \in B\} = 0,$$

*then  $\mu \times \xi_G(B) = 0$ .*

*Proof:* Since  $B$  is universally measurable there exist Borel sets  $A, C \subset K \times G$  with  $A \subset B \subset C$  and  $\mu \times \xi_H \times \xi_G(\psi^{-1}(C \setminus A)) = 0$ . From the assumption on  $B$  and from Fubini's theorem it follows that

$$\begin{aligned} \mu \times \xi_H \times \xi_G(\psi^{-1}(C)) &= \mu \times \xi_H \times \xi_G(\psi^{-1}(A)) = \\ &= \int \int \mu\{x : (x, h, V) \in \psi^{-1}(A)\} d\xi_H(h) d\xi_G(V) \leq \\ &\leq \int \int \mu\{x : (x, h, V) \in \psi^{-1}(B)\} d\xi_H(h) d\xi_G(V) = 0. \end{aligned}$$

Now from Fubini's theorem, from the definition of  $\xi_G$  given in (2.1), and from Lemma 7, it follows that

$$\begin{aligned} 0 &= \mu \times \xi_H \times \xi_G(\psi^{-1}(C)) = \\ &= \int \int \xi_H\{h : (x, h, V) \in \psi^{-1}(C)\} d\xi_G(V) d\mu(x) = \\ &= \int \int \xi_H\{h : (x, h, gU) \in \psi^{-1}(C)\} d\xi_O(g) d\mu(x) = \\ &= \int \int \xi_H\{h : (x, hgU) \in C\} d\xi_O(g) d\mu(x) = \\ &= \int \int \xi_H(\{h : (x, hU) \in C\} \cdot g^{-1}) d\xi_O(g) d\mu(x) = \\ &= \int \xi_O\{g : (x, gU) \in C\} d\mu(x) = \\ &= \int \xi_G\{V : (x, V) \in C\} d\mu(x) = \mu \times \xi_G(C) \geq \mu \times \xi_G(B), \end{aligned}$$

which completes the proof of the lemma.  $\square$

**3.3. Proofs of Theorems 1 and 4.** Fix  $V \in \mathcal{V}$  for the remainder of this section, set  $F = F_V$ , and for each  $h \in H$  set  $V^h = hV$  and  $P_h = P_{(V^h)^\perp}$ . Set

$$Q = \{(x, h) \in X : F(x, h) \neq \Theta^{*m}(P_h \mu, P_h(x)) \text{ or } F(x, h) = \infty \text{ or } F(x, h) = 0\}$$



where  $\Theta^{*m}$  is as defined in (2.2), then  $Q$  is a Borel set. From theorem 2.12 in [M1] it follows that

$$\mu\{x \in K : (x, h) \in Q\} = 0 \text{ for each } h \in H \text{ with } P_h\mu \ll \mathcal{H}^m,$$

hence since  $V \in \mathcal{V}$  we have

$$(3.2) \quad \mu \times \xi_H(Q) = \int_H \mu\{x : (x, h) \in Q\} d\xi_H(h) = 0.$$

Let  $D$  be the set of all  $(x, h) \in X$  such that  $P_h\mu \ll \mathcal{H}^m$ ,  $\mu_{V^h, x}$  is defined,

$$\mu_{V^h, x}(K_w) = \lim_{\epsilon \downarrow 0} \frac{\mu(K_w \cap P_h^{-1}(B(P_h x, \epsilon)))}{P_h\mu(B(P_h x, \epsilon))} \text{ for each } w \in \Lambda^*,$$

and

$$0 < F(x, h) = \lim_{\epsilon \downarrow 0} \frac{P_h\mu(B(P_h(x), \epsilon))}{(2\epsilon)^m} < \infty.$$

From the choice of  $V$ , from Lemma 9 and from (3.2), it follows that  $\mu \times \xi_H(X \setminus D) = 0$ . Set  $D_0 = \cap_{j=0}^{\infty} T^{-j}D$ , then  $\mu \times \xi_H(X \setminus D_0) = 0$  since  $T$  is measure preserving. The following lemma will be used several times below.

**Lemma 13.** *Given  $k \geq 1$  and  $(x, h) \in D_0$ , we have*

$$\mu_{V^h, x}(K_{w_k(x)}) = (F(x, h))^{-1} \cdot r_{w_k(x)}^{s-m} \cdot F(T^k(x, h)).$$

*Proof:* Set  $u = w_k(x)$ , then

$$\begin{aligned} \mu_{V^h, x}(K_u) &= \lim_{\epsilon \downarrow 0} \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{P_h\mu(B(P_h x, \epsilon))} = \\ &= \lim_{\epsilon \downarrow 0} \frac{(2\epsilon)^m}{P_h\mu(B(P_h x, \epsilon))} \cdot \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{(2\epsilon)^m} = \\ &= (F(x, h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(K_u \cap P_h^{-1}(B(P_h x, \epsilon)))}{(2\epsilon)^m}. \end{aligned}$$

For each  $\epsilon > 0$  set  $E_\epsilon = P_{h_u^{-1}h}^{-1}(B(P_{h_u^{-1}h}(\varphi_u^{-1}(x)), \epsilon \cdot r_u^{-1}))$ , then since

$$\begin{aligned} P_h^{-1}(B(P_h x, \epsilon)) &= x + V^h + B(0, \epsilon) = \\ &= \varphi_u(\varphi_u^{-1}(x) + V^{h_u^{-1}h} + B(0, \epsilon \cdot r_u^{-1})) = \varphi_u(E_\epsilon), \end{aligned}$$

it follows that

$$\begin{aligned} \mu_{V^h, x}(K_u) &= (F(x, h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(\varphi_u(K \cap E_\epsilon))}{(2\epsilon)^m} = \\ &= (F(x, h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{1}{(2\epsilon)^m} \sum_{w \in \Lambda^k} p_w \cdot \mu(\varphi_w^{-1}(\varphi_u(K \cap E_\epsilon))). \end{aligned}$$

Given  $w \in \Lambda^k \setminus \{u\}$  we have  $\varphi_u(K) \cap \varphi_w(K) = \emptyset$ , so  $\varphi_w^{-1}(\varphi_u(K)) \cap K = \emptyset$ , and so

$$\begin{aligned}\mu_{V^h, x}(K_u) &= (F(x, h))^{-1} \cdot \lim_{\epsilon \downarrow 0} \frac{p_u \cdot \mu(K \cap E_\epsilon)}{(2\epsilon)^m} = (F(x, h))^{-1} \cdot r_u^{s-m} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(E_\epsilon)}{(2\epsilon \cdot r_u^{-1})^m} = \\ &= (F(x, h))^{-1} \cdot r_u^{s-m} \cdot F(\varphi_u^{-1}(x), h_u^{-1}h) = (F(x, h))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h)),\end{aligned}$$

which proves the lemma.  $\square$

*Proof of theorem 1, part (i):* Assume that  $V$  is such that  $\|F\|_{L^\infty(\mu \times \xi_H)} < \infty$ . Set  $M = \|F\|_{L^\infty(\mu \times \xi_H)}$ ,  $E = \{(x, h) : F(x, h) \leq M\}$  and  $E_1 = D_0 \cap (\cap_{j=0}^\infty T^{-j}(E))$ , then  $\mu \times \xi_H(X \setminus E_1) = 0$ . For  $\xi_H$ -a.e.  $h \in H$  we have

$$\mu\{x \in K : (x, h) \notin E_1\} = 0,$$

fix such  $h_0 \in H$ . For each  $l \geq 1$  set

$$A_l = \{x \in K : (x, h_0) \in E_1 \text{ and } F(x, h_0) \geq l^{-1}\},$$

and fix  $l_0 \geq 1$ . Set  $\kappa = \min\{r_\lambda : \lambda \in \Lambda\}$ , it will now be shown that

$$(3.3) \quad \Theta^{*s-m}(\mu_{V^{h_0}, x}, x) \leq (2\rho\kappa)^{m-s} l_0 M \text{ for each } x \in A_{l_0},$$

where  $\rho$  is as defined in (3.1). Let  $x \in A_{l_0}$  and let  $\kappa\rho > \delta > 0$ . Let  $k \geq 1$  be such that  $r_{w_k(x)} \geq \frac{\delta}{\rho} > r_{w_{k+1}(x)}$ , and set  $u = w_k(x)$ . From Lemma 13 and from  $T^k(x, h_0) \in E$  we get that

$$\mu_{V^{h_0}, x}(K_u) = (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \leq l_0 \cdot r_u^{s-m} \cdot M,$$

and so

$$\begin{aligned}\frac{\mu_{V^{h_0}, x}(B(x, \delta))}{(2\delta)^{s-m}} &\leq \frac{\mu_{V^{h_0}, x}(B(x, \rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_{k+1}(x)})^{s-m}} \leq \\ &\leq \frac{\mu_{V^{h_0}, x}(K_u)}{(2\rho\kappa \cdot r_u)^{s-m}} \leq \frac{l_0 r_u^{s-m} M}{(2\rho\kappa \cdot r_u)^{s-m}} = (2\rho\kappa)^{m-s} l_0 M,\end{aligned}$$

which proves (3.3).

It holds that

$$\{x \in K : (x, h_0) \in E_1\} = \cup_{l=1}^\infty A_l,$$

hence

$$0 = \mu(K \setminus \cup_{l=1}^\infty A_l) = \int \mu_{V^{h_0}, x}(K \setminus \cup_{l=1}^\infty A_l) d\mu(x),$$

and so for  $\mu$ -a.e.  $x \in K$  there exist  $l_x \geq 1$  with  $\mu_{V^{h_0}, x}(A_{l_x} \cap V_x^{h_0}) = \mu_{V^{h_0}, x}(A_{l_x}) > 0$ . Fix such  $x_0 \in K$  and let  $y \in A_{l_{x_0}} \cap V_{x_0}^{h_0}$ , then from (3.3) we get that

$$\Theta^{*s-m}(\mu_{V^{h_0}, x_0}, y) = \Theta^{*s-m}(\mu_{V^{h_0}, y}, y) \leq (2\rho\kappa)^{m-s} l_{x_0} M,$$

and so from Theorem 6.9 in [M1] it follows that

$$\begin{aligned}\mathcal{H}^{s-m}(K \cap V_{x_0}^{h_0}) &\geq \mathcal{H}^{s-m}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \geq \\ &\geq 2^{-(s-m)}(2\rho\kappa)^{s-m}l_{x_0}^{-1}M^{-1} \cdot \mu_{V^{h_0},x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) > 0.\end{aligned}$$

This proves that if  $\|F_V\|_{L^\infty(\mu \times \xi_H)} < \infty$ , then for  $\xi_H$ -a.e.  $h \in H$  we have

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \text{ for } \mu\text{-a.e. } x \in K,$$

and so (i) follows from Lemma 10 and Fubini's theorem.

*Proof of part (ii):* Assume that  $V$  is such that  $\|F\|_{L^\infty(\mu \times \xi_H)} = \infty$ , then

$$\mu \times \xi_H\{(x, h) : F(x, h) > M\} > 0 \text{ for each } 0 < M < \infty.$$

For each integer  $M \geq 1$  set

$$E_M = \{(x, h) \in X : F(x, h) > M\} \text{ and } E_{0,M} = \cap_{N=1}^\infty \cup_{j=N}^\infty T^{-j}(E_M),$$

then  $\mu \times \xi_H(E_M) > 0$ , and so  $\mu \times \xi_H(X \setminus E_{0,M}) = 0$  since  $\mu \times \xi_H$  is ergodic (see Theorem 1.5 in [W]). Set  $\tilde{E} = D_0 \cap (\cap_{M=1}^\infty E_{0,M})$ , then  $\mu \times \xi_H(X \setminus \tilde{E}) = 0$ . For  $\xi_H$ -a.e.  $h \in H$  it holds that  $\mu\{x \in K : (x, h) \notin \tilde{E}\} = 0$ , fix such  $h_0 \in H$  and set

$$A = \{x \in K : (x, h_0) \in \tilde{E}\}.$$

Note that since  $(x, h_0) \in D_0$  for some  $x \in K$ , it follows that  $P_{h_0}\mu \ll \mathcal{H}^m$ . It will now be shown that

$$(3.4) \quad \Theta^{*s-m}(\mu_{V^{h_0},x}, x) = \infty \text{ for each } x \in A.$$

Let  $x \in A$ ,  $M \geq 1$  and  $N \geq 1$  be given, then there exists  $k \geq N$  with  $T^k(x, h_0) \in D_0 \cap E_M$ , and so  $F(T^k(x, h_0)) > M$ . Set  $u = w_k(x)$  and  $\beta = (F(x, h_0))^{-1}$ , then from Lemma 13

$$\mu_{V^{h_0},x}(K_u) = \beta \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \geq \beta \cdot r_u^{s-m} \cdot M.$$

Set  $d = \sup\{|y_1 - y_2| : y_1, y_2 \in K\}$ , then

$$\frac{\mu_{V^{h_0},x}(B(x, d \cdot r_{w_k(x)}))}{(2d \cdot r_{w_k(x)})^{s-m}} \geq \frac{\mu_{V^{h_0},x}(K_u)}{(2d \cdot r_u)^{s-m}} \geq \frac{\beta \cdot r_u^{s-m} \cdot M}{(2d \cdot r_u)^{s-m}} = \frac{M\beta}{(2d)^{s-m}}.$$

Since  $\lim_{k \rightarrow \infty} r_{w_k(x)} = 0$  we get that  $\Theta^{*s-m}(\mu_{V^{h_0},x}, x) \geq \frac{M\beta}{(2d)^{s-m}}$ , and so (3.4) follows since  $M$  can be chosen arbitrarily large.

Let  $x \in A$  and  $y \in A \cap V_x^{h_0}$ , then from (3.4) we get

$$\Theta^{*s-m}(\mu_{V^{h_0},x}, y) = \Theta^{*s-m}(\mu_{V^{h_0},y}, y) = \infty.$$

Now from Theorem 6.9 in [M1] it follows that for each  $M \geq 1$

$$\mathcal{H}^{s-m}(A \cap V_x^{h_0}) \leq M^{-1} \cdot \mu_{V^{h_0},x}(A \cap V_x^{h_0}) \leq M^{-1},$$

and so  $\mathcal{H}^{s-m}(A \cap V_x^{h_0}) = 0$  since  $M$  can be chosen arbitrarily large. Also, from  $\mu(K \setminus A) = 0$  and Theorem 7.7 in [M1] we get that

$$\begin{aligned} \int_{(V^{h_0})^\perp} \mathcal{H}^{s-m}((K \setminus A) \cap V_y^{h_0}) d\mathcal{H}^m(y) &\leq \\ &\leq \text{const} \cdot \mathcal{H}^s(K \setminus A) = \text{const} \cdot \mu(K \setminus A) = 0. \end{aligned}$$

This shows that  $\mathcal{H}^{s-m}((K \setminus A) \cap V_y^{h_0}) = 0$  for  $\mathcal{H}^m$ -a.e.  $y \in (V^{h_0})^\perp$ , and so  $\mathcal{H}^{s-m}((K \setminus A) \cap V_x^{h_0}) = 0$  for  $\mu$ -a.e.  $x \in K$  since  $P_{h_0}\mu \ll \mathcal{H}^m$ . It follows that for  $\mu$ -a.e.  $x \in A$  (and so for  $\mu$ -a.e.  $x \in K$ ) we have

$$\mathcal{H}^{s-m}(K \cap V_x^{h_0}) = \mathcal{H}^{s-m}(A \cap V_x^{h_0}) + \mathcal{H}^{s-m}((K \setminus A) \cap V_x^{h_0}) = 0.$$

From this, Lemma (10) and Fubini's theorem, it follows that  $\mathcal{H}^{s-m}(K \cap V_x^h) = 0$  for  $\mu \times \xi_H$ -a.e.  $(x, h) \in K \times H$ , which proves (ii).

*Proof of part (iii):* Assume that  $\|F_V\|_\infty < \infty$  for  $\xi_G$ -a.e.  $V \in G$ . From Lemma 8 and part (i), it follows that for  $\xi_G$ -a.e.  $V \in G$  it holds for  $\xi_H$ -a.e.  $h \in H$  that

$$\mathcal{H}^{s-m}(K \cap (x + hV)) > 0 \text{ for } \mu\text{-a.e. } x \in K.$$

Set

$$B = \{(x, V) \in K \times G : \mathcal{H}^{s-m}(K \cap V_x) = 0\},$$

then from Lemma 10 we get that  $B$  is a Borel set (hence universally measurable), and so  $\mu \times \xi_G(B) = 0$  by Lemma 12.

For the other direction, set  $\mathcal{W} = \{V \in G : \|F_V\|_\infty = \infty\}$  and assume that  $\xi_G(\mathcal{W}) > 0$ . From part (ii) it follows that for  $\xi_G$ -a.e.  $V \in \mathcal{W}$  we have

$$\mathcal{H}^{s-m}(K \cap (x + hV)) = 0 \text{ for } \mu \times \xi_H\text{-a.e. } (x, h) \in X,$$

and so from Lemma 7

$$\begin{aligned}
0 < \xi_G(\mathcal{W}) &\leq \int \mu \times \xi_H\{(x, h) : \mathcal{H}^{s-m}(K \cap (x + hV)) = 0\} d\xi_G(V) = \\
&= \int \int \xi_H\{h : \mathcal{H}^{s-m}(K \cap (x + hgU)) = 0\} d\xi_O(g) d\mu(x) = \\
&= \int \int \xi_H(\{h : \mathcal{H}^{s-m}(K \cap (x + hU)) = 0\} \cdot g^{-1}) d\xi_O(g) d\mu(x) = \\
&= \int \xi_O\{g : \mathcal{H}^{s-m}(K \cap (x + gU)) = 0\} d\mu(x) = \\
&= \int \xi_G\{V : \mathcal{H}^{s-m}(K \cap V_x) = 0\} d\mu(x) = \\
&= \mu \times \xi_G\{(x, V) : \mathcal{H}^{s-m}(K \cap V_x) = 0\},
\end{aligned}$$

which completes the proof of (iii).

Part (iv) can be proven in a similar manner, and so the proof of Theorem 1 is complete.  $\square$

*Proof of theorem 4, part (i):* Let  $M > 0$  be so large such that for

$$E = \{(x, h) \in X : F(x, h) \leq M\}$$

we have  $\mu \times \xi_H(E) > 0$ . Set  $E_0 = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E)$ , then  $\mu \times \xi_H(X \setminus E_0) = 0$  since  $\mu \times \xi_H$  is ergodic. Set  $E_1 = E_0 \cap D_0$ , then  $\mu \times \xi_H(X \setminus E_1) = 0$ . For  $\xi_H$ -a.e.  $h \in H$  it holds that  $\mu\{x \in K : (x, h) \notin E_1\} = 0$ , fix such  $h_0 \in H$ . For each  $l \geq 1$  set

$$A_l = \{x \in K : (x, h_0) \in E_1 \text{ and } F(x, h_0) \geq l^{-1}\},$$

and fix  $l_0 \geq 1$ . It will now be shown that

$$(3.5) \quad \Theta_*^{s-m}(\mu_{V^{h_0}, x}, x) \leq (2\rho)^{m-s} l_0 M \text{ for each } x \in A_{l_0}.$$

Let  $x \in A_{l_0}$  and let  $N \geq 1$  be given, then since  $(x, h_0) \in E_1$  it follows that there exist  $k \geq N$  with  $T^k(x, h_0) \in E \cap D_0$ , and so  $F(T^k(x, h_0)) \leq M$ . Set  $u = w_k(x)$ , then from Lemma 13 we have

$$\mu_{V^{h_0}, x}(K_u) = (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \leq l_0 r_u^{s-m} M,$$

from which it follows that

$$\frac{\mu_{V^{h_0}, x}(B(x, \rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_k(x)})^{s-m}} \leq \frac{\mu_{V^{h_0}, x}(K_u)}{(2\rho \cdot r_u)^{s-m}} \leq \frac{l_0 r_u^{s-m} M}{(2\rho \cdot r_u)^{s-m}} = (2\rho)^{m-s} l_0 M.$$

This proves (3.5) since  $r_{w_k(x)}$  tends to 0 as  $k$  tends to  $\infty$ .

As in the proof of part (i) of Theorem 1, from  $\mu(K \setminus \cup_{l=1}^{\infty} A_l) = 0$  it follows that for  $\mu$ -a.e.  $x \in K$  there exists  $l_x \geq 1$  with  $\mu_{V^{h_0}, x}(A_{l_x} \cap V_x^{h_0}) > 0$ . Fix such an  $x_0$

and let  $y \in A_{l_{x_0}} \cap V_{x_0}^{h_0}$ , then from (3.5) we get

$$\Theta_*^{s-m}(\mu_{V^{h_0}, x_0}, y) = \Theta_*^{s-m}(\mu_{V^{h_0}, y}, y) \leq (2\rho)^{m-s} l_{x_0} M,$$

and so from Theorem 6.11 in [M1] it follows that

$$\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) \geq \mathcal{P}^{s-m}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) \geq (2\rho)^{s-m} l_{x_0}^{-1} M^{-1} \cdot \mu_{V^{h_0}, x_0}(A_{l_{x_0}} \cap V_{x_0}^{h_0}) > 0.$$

Since  $\xi_G(G \setminus \mathcal{V}) = 0$ , this shows that for  $\xi_G$ -a.e.  $V \in G$  it holds for  $\xi_H$ -a.e.  $h \in H$  that  $\mathcal{P}^{s-m}(K \cap (x + hV)) > 0$  for  $\mu$ -a.e.  $x \in K$ . Set

$$B = \{(x, V) \in K \times G : \mathcal{P}^{s-m}(K \cap V_x) = 0\},$$

then from Lemma 11 we get that  $B$  is universally measurable, and so the claim stated in (i) follows from Lemma 12.

*Proof of part (ii):* Assume  $V$  is such that  $\|\frac{1}{F}\|_{L^\infty(\mu \times \xi_H)} = \infty$ , then

$$\mu \times \xi_H\{(x, h) : F(x, h) < M^{-1}\} > 0 \text{ for each } 0 < M < \infty.$$

For each integer  $M \geq 1$  set

$$E_M = \{(x, h) : F(x, h) < M^{-1}\} \text{ and } E_{0,M} = \bigcap_{N=1}^\infty \bigcup_{j=N}^\infty T^{-j}(E_M),$$

then since  $\mu \times \xi_H$  is ergodic and  $\mu \times \xi_H(E_M) > 0$  it follows that  $\mu \times \xi_H(X \setminus E_{0,M}) = 0$ . Set  $\tilde{E} = D_0 \cap (\bigcap_{M=1}^\infty E_{0,M})$ , then  $\mu \times \xi_H(X \setminus \tilde{E}) = 0$ . For  $\xi_H$ -a.e.  $h \in H$  it holds that  $\mu\{x \in K : (x, h) \notin \tilde{E}\} = 0$ , fix such  $h_0 \in H$  and set  $A = \{x \in K : (x, h_0) \in \tilde{E}\}$ . It will now be shown that

$$(3.6) \quad \Theta_*^{s-m}(\mu_{V^{h_0}, x}, x) = 0 \text{ for each } x \in A.$$

Let  $x \in A$ ,  $M \geq 1$  and  $N \geq 1$  be given, then there exists  $k \geq N$  with  $T^k(x, h_0) \in D_0 \cap E_M$ , and so  $F(T^k(x, h_0)) < M^{-1}$ . Set  $u = w_k(x)$ , then from Lemma 13

$$\mu_{V^{h_0}, x}(K_u) = (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot F(T^k(x, h_0)) \leq (F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot M^{-1},$$

from which it follows that

$$\begin{aligned} \frac{\mu_{V^{h_0}, x}(B(x, \rho \cdot r_{w_k(x)}))}{(2\rho \cdot r_{w_k(x)})^{s-m}} &\leq \frac{\mu_{V^{h_0}, x}(K_u)}{(2\rho \cdot r_u)^{s-m}} \leq \\ &\leq \frac{(F(x, h_0))^{-1} \cdot r_u^{s-m} \cdot M^{-1}}{(2\rho \cdot r_u)^{s-m}} = (2\rho)^{m-s} \cdot (F(x, h_0))^{-1} \cdot M^{-1}. \end{aligned}$$

This shows that

$$\Theta_*^{s-m}(\mu_{V^{h_0}, x}, x) \leq (2\rho)^{m-s} \cdot (F(x, h_0))^{-1} \cdot M^{-1},$$

and so (3.6) holds since  $M$  can be chosen arbitrarily large.

We have

$$0 = \mu(K \setminus A) = \int \mu_{V^{h_0}, x}(K \setminus A) d\mu(x),$$

hence  $\mu_{V^{h_0}, x}(A \cap V_x^{h_0}) > 0$  for  $\mu$ -a.e.  $x \in K$ . Fix such  $x_0 \in K$  and let  $y \in A \cap V_{x_0}^{h_0}$ , then from (3.6) we get

$$\Theta_*^{s-m}(\mu_{V^{h_0}, x_0}, y) = \Theta_*^{s-m}(\mu_{V^{h_0}, y}, y) = 0.$$

Now from Theorem 6.11 in [M1] it follows that for each  $\epsilon > 0$

$$\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) \geq \mathcal{P}^{s-m}(A \cap V_{x_0}^{h_0}) \geq \epsilon^{-1} \cdot \mu_{V^{h_0}, x_0}(A \cap V_{x_0}^{h_0}),$$

which shows that  $\mathcal{P}^{s-m}(K \cap V_{x_0}^{h_0}) = \infty$  since  $\epsilon$  can be chosen arbitrarily small and  $\mu_{V^{h_0}, x_0}(A \cap V_{x_0}^{h_0}) > 0$ .

This proves that if  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$ , then for  $\xi_H$ -a.e.  $h \in H$  we have  $\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty$  for  $\mu$ -a.e.  $x \in K$ , and so (ii) follows from Lemma (11) and Fubini's theorem.

*Proof of part (iii):* Assume that  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$  for  $\xi_G$ -a.e.  $V \in G$ , then from Lemma 8 and part (ii) it follows that for  $\xi_G$ -a.e.  $V \in G$  it holds for  $\xi_H$ -a.e.  $h \in H$  that  $\mathcal{P}^{s-m}(K \cap (x + hV)) = \infty$  for  $\mu$ -a.e.  $x \in K$ . Set

$$B = \{(x, V) \in K \times G : \mathcal{P}^{s-m}(K \cap V_x) < \infty\},$$

then from Lemma 11 we get that  $B$  is universally measurable, and so the claim stated in (iii) follows from Lemma 12. This completes the proof of Theorem 4.  $\square$

**3.4. Proofs of Corollaries 2, 3 and 5.** The following lemma will be used in the proofs of Corollaries 2 and 5. For its proof see Lemma 3.2 in [FK] and the discussion before it.

**Lemma 14.** *Assume  $m = 1$  and  $s > 2$ , then  $P_{V^\perp} \mu \ll \mathcal{H}^m$  and  $\frac{dP_{V^\perp} \mu}{d\mathcal{H}^m}$  has a continuous version for  $\xi_G$ -a.e.  $V \in G$ .*

*Proof of corollary 2:* Assuming  $m = 1$ ,  $s > 2$  and  $|H| < \infty$ , it will be shown that  $\|F_V\|_{L^\infty(\mu \times \xi_H)} < \infty$  for  $\xi_G$ -a.e.  $V \in G$ . From this and from part (iii) of Theorem 1 the corollary will follow. Set

$$E = \{V \in G : P_{V^\perp} \mu \ll \mathcal{H}^m \text{ and } \frac{dP_{V^\perp} \mu}{d\mathcal{H}^m} \text{ is continuous}\},$$

then from Lemma 14 we get  $\xi_G(G \setminus E) = 0$ . From this and from Lemma 7 it now follows that

$$\begin{aligned} 0 &= \xi_G(G \setminus E) = \xi_O\{g \in O(n) : gU \notin E\} = \\ &= \int \xi_H\{h : hgU \notin E\} d\xi_O(g) = \int \xi_H\{h : hV \notin E\} d\xi_G(V), \end{aligned}$$

and so  $\xi_H\{h : hV \notin E\} = 0$  for  $\xi_G$ -a.e.  $V$ . We fix such a  $V \in G$ . Since  $|H| < \infty$ , for each  $h \in H$  we have  $\xi_H\{h\} > 0$ , and so  $hV \in E$ .

For each  $h \in H$  and  $y \in (hV)^\perp$  set  $Q_h(y) = \Theta_*^m(P_{(hV)^\perp}\mu, y)$ , fix  $h_0 \in H$ , and set  $W = (h_0V)^\perp$ . Since  $\mathcal{H}^m(B(y, r) \cap W) = (2\epsilon)^m$  for each  $y \in W$  and  $0 < \epsilon < \infty$ , it follows from Theorem 2.12 in [M1] that  $Q_{h_0}(y) = \frac{dP_W\mu}{d\mathcal{H}^m}(y)$  for  $\mathcal{H}^m$ -a.e.  $y \in W$ , i.e. the function  $Q_{h_0}$  equals a continuous function as members of  $L^1(W, \mathcal{H}^m)$ . Also, since  $\mu$  is supported on a compact set it follows that the set  $\{y \in W : Q_{h_0}(y) \neq 0\}$  is bounded, so  $Q_{h_0}$  equals a continuous function with compact support in  $L^1(W, \mathcal{H}^m)$ , which shows that  $\|Q_{h_0}\|_{L^\infty(W, \mathcal{H}^m)} < \infty$ . Since  $P_W\mu \ll \mathcal{H}^m$  it follows that  $\|Q_{h_0}\|_{L^\infty(P_W\mu)} < \infty$ .

Now set  $M = \max\{\|Q_h\|_{L^\infty(P_{(hV)^\perp}\mu)} : h \in H\}$ , then  $M < \infty$  since  $|H| < \infty$ . Also, we have

$$\begin{aligned} 0 &= \frac{1}{|H|} \sum_{h \in H} P_{(hV)^\perp}\mu\{y \in (hV)^\perp : |Q_h(y)| > M\} = \\ &= \frac{1}{|H|} \sum_{h \in H} \mu\{x \in K : |Q_h(P_{(hV)^\perp}(x))| > M\} = \\ &= \frac{1}{|H|} \sum_{h \in H} \mu\{x \in K : |F_V(x, h)| > M\} = \\ &= \int \mu\{x \in K : |F_V(x, h)| > M\} d\xi_H(h) = \\ &= \mu \times \xi_H\{(x, h) \in K \times H : |F_V(x, h)| > M\}, \end{aligned}$$

which shows that  $\|F_V\|_{L^\infty(\mu \times \xi_H)} \leq M < \infty$ . This completes the proof of corollary 2.  $\square$

*Proof of corollary 3:* Assume that  $H = O(n)$  and

$$\mu \times \xi_G\{(x, V) : \mathcal{H}^{s-m}(K \cap V_x) > 0\} > 0.$$

Let  $V \in \mathcal{V}$ , then since  $\xi_H = \xi_O$  we have

$$\mu \times \xi_H\{(x, h) : \mathcal{H}^{s-m}(K \cap (x + hV)) > 0\} > 0,$$



and so from part (ii) of theorem 1 it follows that  $\|F_V\|_{L^\infty(\mu \times \xi_H)} < \infty$ . Set  $M = \|F_V\|_{L^\infty(\mu \times \xi_H)}$ , set

$$E = \{W \in G : P_{W^\perp} \mu \ll \mathcal{H}^m \text{ and } \left\| \frac{dP_{W^\perp} \mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} \leq M\},$$

and for each  $h \in H$  set  $P_h = P_{(hV)^\perp}$ .

We shall first show that  $\xi_G(G \setminus E) = 0$ . Since  $P_{W^\perp} \mu \ll \mathcal{H}^m$  for  $\xi_G$ -a.e.  $W \in G$  (see the proof of lemma 8), and since  $\xi_H = \xi_O$ , we have

$$\begin{aligned} (3.7) \quad \xi_G(G \setminus E) &= \xi_G(G \setminus \{W \in G : P_{W^\perp} \mu \ll \mathcal{H}^m\}) + \\ &\quad + \xi_G\{W \in G : P_{W^\perp} \mu \ll \mathcal{H}^m \text{ and } \left\| \frac{dP_{W^\perp} \mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} > M\} = \\ &= \xi_H\{h : P_h \mu \ll \mathcal{H}^m \text{ and } \left\| \frac{dP_h \mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} > M\}. \end{aligned}$$

Let  $h \in H$  be such that  $P_h \mu \ll \mathcal{H}^m$  and  $\left\| \frac{dP_h \mu}{d\mathcal{H}^m} \right\|_{L^\infty(P_h \mu)} \leq M$ , then

$$\begin{aligned} 0 &= P_h \mu\{y \in (hV)^\perp : \frac{dP_h \mu}{d\mathcal{H}^m}(y) > M\} = \\ &= \int_{(hV)^\perp} 1_{\{\frac{dP_h \mu}{d\mathcal{H}^m} > M\}} \cdot \frac{dP_h \mu}{d\mathcal{H}^m} d\mathcal{H}^m \geq \\ &\geq M \cdot \mathcal{H}^m\{y \in (hV)^\perp : \frac{dP_h \mu}{d\mathcal{H}^m}(y) > M\}, \end{aligned}$$

which shows that  $\left\| \frac{dP_h \mu}{d\mathcal{H}^m} \right\|_{L^\infty(\mathcal{H}^m)} \leq M$ . From this and from (3.7) it follows that

$$(3.8) \quad \xi_G(G \setminus E) = \xi_H\{h : P_h \mu \ll \mathcal{H}^m \text{ and } \left\| \frac{dP_h \mu}{d\mathcal{H}^m} \right\|_{L^\infty(P_h \mu)} > M\}.$$

From Theorem 2.12 in [M1] we get that for each  $h \in H$  with  $P_h \mu \ll \mathcal{H}^m$

$$F_V(x, h) = \frac{dP_h \mu}{d\mathcal{H}^m}(P_h(x)) \text{ for } \mu\text{-a.e. } x \in K,$$

and so from (3.8)

$$\begin{aligned} \xi_G(G \setminus E) &\leq \xi_H\{h : \|F_V(\cdot, h)\|_{L^\infty(\mu)} > M\} = \\ &= \xi_H\{h : \mu\{x : F_V(x, h) > \|F_V\|_{L^\infty(\mu \times \xi_H)}\} > 0\} = 0. \end{aligned}$$

Since  $\xi_G(\mathcal{W}) > 0$  for every non-empty open set  $\mathcal{W} \subset G$ , it follows from  $\xi_G(G \setminus E) = 0$  that  $E$  is dense in  $G$ , and so in order to prove the corollary it suffice to show that  $E$  is a closed subset of  $G$ . Let  $W_0 \in \overline{E}$ , let  $y \in W_0^\perp$  and let  $r \in (0, \infty)$ . Given  $\epsilon > 0$  there exists  $W \in E$  so close to  $W_0$  in  $G$  (with respect to the metric  $d_G$  defined in

section 3.1), such that

$$P_{W_0^\perp}^{-1}(B(y, r)) \cap K \subset P_{W^\perp}^{-1}(B(P_{W^\perp}y, r + \epsilon)).$$

From this and since  $W \in E$  it follows that

$$\begin{aligned} P_{W_0^\perp} \mu(B(y, r)) &= \mu(P_{W_0^\perp}^{-1}(B(y, r)) \cap K) \leq \\ &\leq \mu(P_{W^\perp}^{-1}(B(P_{W^\perp}y, r + \epsilon))) = P_{W^\perp} \mu(B(P_{W^\perp}y, r + \epsilon)) = \\ &= \int_{B(P_{W^\perp}y, r + \epsilon) \cap W^\perp} \frac{dP_{W^\perp} \mu}{d\mathcal{H}^m} d\mathcal{H}^m \leq \\ &\leq M \cdot \mathcal{H}^m(B(P_{W^\perp}y, r + \epsilon) \cap W^\perp) = M \cdot (2 \cdot (r + \epsilon))^m, \end{aligned}$$

and since this holds for each  $\epsilon > 0$  we have

$$P_{W_0^\perp} \mu(B(y, r) \cap W_0^\perp) \leq M \cdot (2r)^m = M \cdot \mathcal{H}^m(B(y, r) \cap W_0^\perp).$$

This holds for every  $y \in W_0^\perp$  and  $r \in (0, \infty)$ , hence  $W_0 \in E$  by Theorem 2.12 in [M1], which shows that  $E$  is closed in  $G$  and completes the proof of the corollary.  $\square$

*Proof of corollary 5:* Assuming  $m = 1$  and  $s > 2$ , it will be shown that  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$  for  $\xi_G$ -a.e.  $V \in G$ . From this and part (iii) of Theorem 4 the corollary will follow. Set

$$E = \{V \in G : P_{V^\perp} \mu \ll \mathcal{H}^m \text{ and } \frac{dP_{V^\perp} \mu}{d\mathcal{H}^m} \text{ is continuous}\},$$

then as in the proof of corollary 2 it follows from Lemma 14 and Lemma 7 that

$$0 = \xi_G(G \setminus E) = \int \xi_H\{h : hV \notin E\} d\xi_G(V),$$

and so  $\xi_H\{h : hV \notin E\} = 0$  for  $\xi_G$ -a.e.  $V$ . Fix such  $V \in G$ , let  $M > 0$ , set

$$A = \{h \in H : hV \in E\},$$

and for each  $h \in H$  and  $y \in (hV)^\perp$  set  $Q_h(y) = \Theta_*^m(P_{(hV)^\perp} \mu, y)$  and

$$L_h = \{y \in (hV)^\perp : 0 < Q_h(y) \leq M^{-1}\}.$$

Fix  $h_0 \in A$  and set  $W = (h_0V)^\perp$ . From Theorem 2.12 in [M1] it follows that  $Q_{h_0}(y) = \frac{dP_W \mu}{d\mathcal{H}^m}(y)$  for  $\mathcal{H}^m$ -a.e.  $y \in W$ , hence the function  $Q_{h_0}$  equals a continuous function in  $L^1(W, \mathcal{H}^m)$ . Also, since  $\mu$  is supported on a compact set, it follows that the set  $\{y \in W : Q_{h_0}(y) \neq 0\}$  is bounded. From these two facts it easily follows that  $\mathcal{H}^m(L_{h_0}) > 0$ , and so  $P_W \mu(L_{h_0}) > 0$  since  $Q_{h_0} = \frac{dP_W \mu}{d\mathcal{H}^m}$  and  $Q_{h_0} > 0$  on  $L_{h_0}$ . From this we get that

$$0 < \mu\{x \in K : Q_{h_0}(P_W(x)) \leq M^{-1}\} = \mu\{x \in K : F_V(x, h_0) \leq M^{-1}\},$$

and so by Fubini's theorem

$$\mu \times \xi_H \left\{ (x, h) : \frac{1}{F_V(x, h)} \geq M \right\} = \int_A \mu \{ x \in K : F_V(x, h) \leq M^{-1} \} d\xi_H(h) > 0.$$

It follows that  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} \geq M$ , and so  $\left\| \frac{1}{F_V} \right\|_{L^\infty(\mu \times \xi_H)} = \infty$  since we can choose  $M$  as large as we want. This completes the proof of the corollary.  $\square$

#### 4. PROOF OF THEOREM 6

Set  $\Lambda = \{1, 2\}$ . Given  $0 < \rho < \frac{1}{2}$ , define  $f_{\rho,1}, f_{\rho,2} : \mathbb{R} \rightarrow \mathbb{R}$  by  $f_{\rho,1}(x) = \rho \cdot x$  and  $f_{\rho,2}(x) = \rho \cdot x + 1 - \rho$  for each  $x \in \mathbb{R}$ , let  $C_\rho \subset [0, 1]$  be the attractor of the IFS  $\{f_{\rho,1}, f_{\rho,2}\}$ , set  $d_\rho = \dim_H C_\rho$  (so that  $d_\rho = \frac{\log 2}{\log \rho^{-1}}$ ), and for each  $E \subset \mathbb{R}$  set  $\mu_\rho(E) = \frac{\mathcal{H}^{d_\rho}(C_\rho \cap E)}{\mathcal{H}^{d_\rho}(C_\rho)}$ . Let  $0 < a < b < \frac{1}{2}$  be such that  $\frac{1}{a}$  and  $\frac{1}{b}$  are Pisot numbers,  $\frac{\log b}{\log a}$  is irrational, and  $d_a + d_b > 1$ . Let  $I = [0, 1)$  and let  $\mathcal{L}$  be Lebesgue measure on  $I$ . Fix  $\tau \in (0, \infty)$ , and for each  $t \in I$  and  $z \in \mathbb{R}^2$  define  $W^t = \{x \cdot (1, \tau \cdot a^t) : x \in \mathbb{R}\}$ ,  $V^t = (W^t)^\perp$  and  $V_z^t = z + V^t$ . In order to prove Theorem 6 we shall first prove the following:

**Theorem 15.** *For  $\mu_a \times \mu_b \times \mathcal{L}$ -a.e.  $(x, y, t) \in C_a \times C_b \times I$  it holds that*

$$\mathcal{H}^{d_a + d_b - 1}((C_a \times C_b) \cap V_{(x,y)}^t) = 0.$$

**4.1. Preliminaries.** Set  $\alpha = \frac{\log b}{\log a}$  (so  $\alpha \in I \setminus \mathbb{Q}$ ), and for each  $t \in I$  set  $R(t) = t + \alpha \pmod{1}$ . Given  $0 < \rho < \frac{1}{2}$  and a word  $\lambda_1 \dots \lambda_l = w \in \Lambda^*$ , write  $f_{\rho,w} = f_{\rho,\lambda_1} \circ \dots \circ f_{\rho,\lambda_l}$  and  $C_{\rho,w} = f_{\rho,w}(C_\rho)$ . For each  $n \geq 1$  and  $x \in C_\rho$  let  $w_{\rho,n}(x) \in \Lambda^n$  be the unique word of length  $n$  which satisfies  $x \in C_{\rho,w_{\rho,n}(x)}$ , and let  $S_\rho(x) = f_{\rho,w_{\rho,1}(x)}^{-1}(x)$ . We also write  $w_{\rho,0}(x) = \emptyset$  and  $C_{\rho,\emptyset} = C_\rho$ .

The following dynamical system will be used in the proof of Theorem 15. The idea of using this system comes from the partition operator introduced in section 10 of [HS]. Set  $K = C_a \times C_b$ ,  $X = K \times I$ ,  $\mu = \mu_a \times \mu_b$ ,  $\nu = \mu \times \mathcal{L}$ , and for each  $(x, y, t) \in X$  define

$$T(x, y, t) = \begin{cases} (x, S_b(y), R(t)) & , \text{ if } t \in [0, 1 - \alpha) \\ ((S_a(x), S_b(y), R(t))) & , \text{ else} \end{cases}.$$

It is easy to check that the system  $(X, \nu, T)$  is measure preserving, and from Lemma 2.2 in [B2] it follows that it is ergodic.

Let  $\mathcal{R}$  be the Borel  $\sigma$ -algebra of  $\mathbb{R}^2$ . For each  $t \in I$  let  $P_t$  be the orthogonal projection onto  $W^t$ , and let  $\{\mu_{t,z}\}_{z \in \mathbb{R}^2}$  be the disintegration of  $\mu$  with respect to  $P_t^{-1}(\mathcal{R})$  (see section 3.1 above). Also, for each  $(z, t) \in X$  define  $F(z, t) = \Theta_*^1(P_t \mu, P_t z)$ .

#### 4.2. Auxiliary lemmas.

**Lemma 16.** *It holds that  $I_1(\mu) < \infty$ , where  $I_1(\mu)$  is the 1-energy of  $\mu$ .*

*Proof:* Set  $\delta = 1 - 2b$ , then for each  $(x, y) \in \mathbb{R}^2$  and  $k \geq 1$

$$\begin{aligned} \mu(B((x, y), \delta \cdot a^k)) &\leq \mu((x - \delta \cdot a^k, x + \delta \cdot a^k) \times (y - \delta \cdot a^k, y + \delta \cdot a^k)) \leq \\ &\leq \mu_a(x - \delta \cdot a^k, x + \delta \cdot a^k) \cdot \mu_b(y - \delta \cdot a^k, y + \delta \cdot a^k) \leq 2^{-k} \cdot 2^{-[k \log_b a]} \leq \\ &\leq 2^{-k} \cdot 2^{1-k \log_b a} = 2 \cdot a^{k(1+\log_b a) \log_a 2^{-1}} = 2 \cdot a^{k(d_a+d_b)}. \end{aligned}$$

This shows that there exists a constant  $M > 0$  with  $\mu(B(z, r)) \leq M \cdot r^{d_a+d_b}$  for each  $z \in \mathbb{R}^2$  and  $r > 0$ . Since  $d_a + d_b > 1$ , the lemma follows from the discussion found at the beginning of chapter 8 of [M1].  $\square$

**Lemma 17.** *Let  $n_1, n_2 \geq 1$ ,  $w_1 \in \Lambda^{n_1}$  and  $w_2 \in \Lambda^{n_2}$ . For each  $(x, y) \in K$  set  $g(x, y) = (f_{a,w_1}(x), f_{b,w_2}(y))$ , then for each Borel set  $B \subset K$*

$$\mu(g(B)) = 2^{-n_1-n_2} \cdot \mu(B).$$

*Proof:* We prove this by using the  $\pi$ - $\lambda$  theorem (see [B1]). Let  $\mathcal{E}$  be the collection of all Borel sets  $B \subset K$  which satisfy  $\mu(g(B)) = 2^{-n_1-n_2} \cdot \mu(B)$ , then  $\mathcal{E}$  is a  $\lambda$ -system. Set

$$\mathcal{P} = \{C_{a,u_1} \times C_{b,u_2} : u_1, u_2 \in \Lambda^*\} \cup \{\emptyset\},$$

then  $\mathcal{P}$  is a  $\pi$ -system,  $\mathcal{P} \subset \mathcal{E}$  and  $\sigma(\mathcal{P})$  equals the collection of all Borel subsets of  $K$ . From the  $\pi$ - $\lambda$  theorem it follows that  $\sigma(\mathcal{P}) \subset \mathcal{E}$ , hence  $\mathcal{E}$  equals the collection of all Borel subsets of  $K$ , and the lemma is proven.  $\square$

**Lemma 18.** *It holds that  $0 < \mathcal{H}^{d_a+d_b}(K) < \infty$ , and  $\mu(E) = \frac{\mathcal{H}^{d_a+d_b}(K \cap E)}{\mathcal{H}^{d_a+d_b}(K)}$  for each Borel set  $E \subset \mathbb{R}^2$ .*

*Proof:* From Theorem 8.10 in [M1] it follows that  $\mathcal{H}^{d_a+d_b}(K) > 0$ , and by an elementary covering argument it can be shown that  $\mathcal{H}^{d_a+d_b}(K) < \infty$ . The rest of the lemma can be proven by using the  $\pi$ - $\lambda$  theorem, as in the proof of Lemma 17.  $\square$

**Lemma 19.** *Let  $0 < M < \infty$  and set  $E_M = \{(z, t) \in X : F(z, t) > M\}$ , then  $\nu(E_M) > 0$ .*

*Proof:* Assume by contradiction that  $\nu(E_M) = 0$  and set

$$L = \{t \in I : \mu\{z : (z, t) \in E_M\} = 0\},$$

then  $\mathcal{L}(I \setminus L) = 0$ , and so  $\overline{L} = I$ . Set

$$A = \{t \in I : P_t \mu \ll \mathcal{H}^1 \text{ and } \left\| \frac{dP_t \mu}{d\mathcal{H}^1} \right\|_{L^\infty(\mathcal{H}^1)} \leq M\},$$

and let  $t \in L$ . For  $P_t\mu$ -a.e.  $z \in W^t$  we have  $\Theta_*^1(P_t\mu, z) \leq M$ , hence from parts (2) and (3) of Theorem 2.12 in [M1] it follows that  $t \in A$ . This shows that  $L \subset A$ , and so that  $\overline{A} = I$ . By an argument similar to the one given at the end of the proof of Corollary 3, it can be shown that  $A$  is a closed subset of  $I$ , and so  $A = I$ . In particular it follows that  $P_t\mu \ll \mathcal{H}^1$  for each  $t \in I$ , which is a contradiction to Theorem 4.1 in [NPS]. This shows that we must have  $\nu(E_M) > 0$ , and the lemma is proven.  $\square$

**4.3. Proofs of Theorems 15 and 6.** *Proof of theorem 15:* Let  $D$  be the set of all  $(z, t) \in X$  such that  $P_t\mu \ll \mathcal{H}^1$ ,  $\mu_{t,z}$  is defined,

$$\mu_{t,z}(C_{a,w_1} \times C_{b,w_2}) = \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_1} \times C_{b,w_2}) \cap P_t^{-1}(B(P_t z, \epsilon)))}{P_t\mu(B(P_t z, \epsilon))}$$

for each  $w_1, w_2 \in \Lambda^*$ , and

$$0 < F(z, t) = \lim_{\epsilon \downarrow 0} \frac{P_t\mu(B(P_t z, \epsilon))}{2\epsilon} < \infty.$$

From Lemma 16 and from the same arguments as the ones given at the beginning of section 3.3, it follows that  $\nu(X \setminus D) = 0$ . Set  $D_0 = \cap_{j=0}^{\infty} T^{-j}D$ , then  $\nu(X \setminus D_0) = 0$  since  $T$  is measure preserving.

For  $0 < M < \infty$  let  $E_M$  be as in Lemma 19, and set  $E_{0,M} = \cap_{N=1}^{\infty} \cup_{j=N}^{\infty} T^{-j}(E_M)$ . Since  $\nu(E_M) > 0$ , it follows from the ergodicity of  $(X, \nu, T)$  that  $\nu(X \setminus E_{0,M}) = 0$ . Set  $D_1 = D_0 \cap (\cap_{M=1}^{\infty} E_{0,M})$ , then  $\nu(X \setminus D_1) = 0$ . For  $\mathcal{L}$ -a.e.  $t \in I$  it holds that  $\mu\{z \in K : (z, t) \notin D_1\} = 0$ , fix such  $t_0 \in I$  and set  $A = \{z \in K : (z, t_0) \in D_1\}$ . Note that from  $A \neq \emptyset$  it follows that  $P_{t_0}\mu \ll \mathcal{H}^1$ .

Set  $\eta = d_a + d_b - 1$ . It will now be shown that

$$(4.1) \quad \Theta^{*\eta}(\mu_{t_0,z}, z) = \infty \text{ for each } z \in A.$$

Let  $(x, y) = z \in A$  and set  $\beta = (F(z, t_0))^{-1}$ , then  $0 < \beta < \infty$  since  $(z, t_0) \in D_0$ . Let  $M \geq 1$  and  $N \geq 1$  be given, then there exists  $k \geq N$  with  $T^k(z, t_0) \in D_0 \cap E_M$ , and so  $F(T^k(z, t_0)) > M$ . Set  $l = [t_0 + k\alpha]$ , then

$$\begin{aligned} (4.2) \quad \mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)}) &= \\ &= \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0} z, \epsilon)))}{P_{t_0}\mu(B(P_{t_0} z, \epsilon))} = \\ &= \lim_{\epsilon \downarrow 0} \frac{2\epsilon}{P_{t_0}\mu(B(P_{t_0} z, \epsilon))} \cdot \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0} z, \epsilon)))}{2\epsilon} = \\ &= \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu((C_{a,w_l(x)} \times C_{b,w_k(y)}) \cap P_{t_0}^{-1}(B(P_{t_0} z, \epsilon)))}{2\epsilon}. \end{aligned}$$

For each  $(x', y') \in \mathbb{R}^2$  set  $g(x', y') = (f_{a, w_l(x)}(x'), f_{b, w_k(y)}(y'))$ , then

$$(4.3) \quad C_{a, w_l(x)} \times C_{b, w_k(y)} = f_{a, w_l(x)}(C_a) \times f_{b, w_k(y)}(C_b) = g(C_a \times C_b).$$

Let  $\epsilon > 0$ , and let  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear map with  $L(1, 0) = (a^l, 0)$  and  $L(0, 1) = (0, b^k)$ . Since  $L$  is the linear part of the affine transformation  $g$ , we have

$$(4.4) \quad \begin{aligned} P_{t_0}^{-1}(B(P_{t_0}z, \epsilon)) &= z + V^{t_0} + B(0, \epsilon) = \\ &= g \circ g^{-1}(z) + L \circ L^{-1}(V^{t_0}) + L \circ L^{-1}(B(0, \epsilon)) = \\ &= g(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}(B(0, \epsilon))). \end{aligned}$$

From  $a^{-l} \geq a^{-t_0-k\alpha+1} \geq a \cdot b^{-k}$ , we obtain

$$(4.5) \quad L^{-1}(B(0, \epsilon)) \supset B(0, \epsilon \cdot a \cdot b^{-k}).$$

Also we have

$$\begin{aligned} L^{-1}(V^{t_0}) &= L^{-1}((W^{t_0})^\perp) = L^{-1}((1, \tau \cdot a^{t_0}) \cdot \mathbb{R})^\perp = \\ &= L^{-1}((\tau \cdot a^{t_0}, -1) \cdot \mathbb{R}) = (\tau \cdot a^{t_0} \cdot a^{-l}, -b^{-k}) \cdot \mathbb{R} = (\tau \cdot a^{t_0} \cdot \frac{b^k}{a^l}, -1) \cdot \mathbb{R}, \end{aligned}$$

and so since

$$\frac{b^k}{a^l} = a^{k \cdot \log_a b - l} = a^{k\alpha - [t_0 + k\alpha]},$$

it follows that

$$(4.6) \quad L^{-1}(V^{t_0}) = (\tau \cdot a^{t_0+k\alpha-[t_0+k\alpha]}, -1) \cdot \mathbb{R} = ((1, \tau \cdot a^{R^k(t_0)}) \cdot \mathbb{R})^\perp = V^{R^k(t_0)}.$$

Set

$$Q_\epsilon = P_{R^k(t_0)}^{-1}(B(P_{R^k(t_0)}(f_{a, w_l(x)}^{-1}(x), f_{b, w_k(y)}^{-1}(y)), \epsilon ab^{-k})),$$

then from (4.4), (4.5) and (4.6) it follows that

$$\begin{aligned} P_{t_0}^{-1}(B(P_{t_0}z, \epsilon)) &= g(g^{-1}(z) + L^{-1}(V^{t_0}) + L^{-1}(B(0, \epsilon))) \supset \\ &\supset g((f_{a, w_l(x)}^{-1}(x), f_{b, w_k(y)}^{-1}(y)) + V^{R^k(t_0)} + B(0, \epsilon ab^{-k})) = g(Q_\epsilon). \end{aligned}$$

Now from (4.2), (4.3) and Lemma 17 we get that

$$\begin{aligned} \mu_{t_0, z}(C_{a, w_l(x)} \times C_{b, w_k(y)}) &= \beta \cdot \lim_{\epsilon \downarrow 0} \frac{\mu(g((C_a \times C_b) \cap Q_\epsilon))}{2\epsilon} = \\ &= \beta \cdot 2^{-l-k} \cdot \frac{a}{b^k} \cdot \lim_{\epsilon \downarrow 0} \frac{\mu((C_a \times C_b) \cap Q_\epsilon)}{2\epsilon ab^{-k}} \geq \\ &\geq \frac{\beta}{2} \cdot 2^{-k-k\alpha} \cdot \frac{a}{b^k} \cdot F((f_{a, w_l(x)}^{-1}(x), f_{b, w_k(y)}^{-1}(y)), R^k(t_0)) = \\ &= \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot F(T^k(z, t_0)) \geq \frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M. \end{aligned}$$

Since

$$C_{a,w_l(x)} \times C_{b,w_k(y)} \subset B(z, \frac{2 \cdot b^k}{a}) \text{ and } 2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} = 1,$$

it follows that

$$\begin{aligned} \frac{\mu_{t_0,z}(B(z, \frac{2 \cdot b^k}{a}))}{(4a^{-1} \cdot b^k)^\eta} &\geq \frac{\mu_{t_0,z}(C_{a,w_l(x)} \times C_{b,w_k(y)})}{(4a^{-1} \cdot b^k)^\eta} \geq \frac{\frac{\beta a}{2} \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot M}{(4a^{-1} \cdot b^k)^\eta} \geq \\ &\geq \frac{\beta a^2}{8} \cdot M \cdot 2^{-k-k\alpha} \cdot b^{-k} \cdot b^{-k\eta} = \frac{\beta a^2}{8} \cdot M. \end{aligned}$$

This shows that  $\Theta^{*\eta}(\mu_{t_0,z}, z) \geq \frac{\beta a^2}{8} \cdot M$ , which proves (4.1) since  $\beta > 0$  and  $M$  can be chosen arbitrarily large.

Let  $z \in A$  and  $u \in A \cap V_z^{t_0}$ , then from (4.1)

$$\Theta^{*\eta}(\mu_{t_0,z}, u) = \Theta^{*\eta}(\mu_{t_0,u}, u) = \infty,$$

and so from Theorem 6.9 in [M1] we get that  $\mathcal{H}^\eta(A \cap V_z^{t_0}) = 0$ . Also it holds that  $\mu(K \setminus A) = 0$ , hence from Theorem 7.7 in [M1] and from Lemma 18 we get that

$$\int_{W^{t_0}} \mathcal{H}^\eta((K \setminus A) \cap V_u^{t_0}) d\mathcal{H}^1(u) \leq \text{const} \cdot \mathcal{H}^{\eta+1}(K \setminus A) = \text{const} \cdot \mu(K \setminus A) = 0.$$

This shows that  $\mathcal{H}^\eta((K \setminus A) \cap V_u^{t_0}) = 0$  for  $\mathcal{H}^1$ -a.e.  $u \in W^{t_0}$ , and so  $\mathcal{H}^\eta((K \setminus A) \cap V_z^{t_0}) = 0$  for  $\mu$ -a.e.  $z \in K$  since  $P_{t_0}\mu \ll \mathcal{H}^1$ . It follows that for  $\mu$ -a.e.  $z \in A$ , and so for  $\mu$ -a.e.  $z \in K$ ,

$$\mathcal{H}^\eta(K \cap V_z^{t_0}) = \mathcal{H}^\eta(A \cap V_z^{t_0}) + \mathcal{H}^\eta((K \setminus A) \cap V_z^{t_0}) = 0.$$

From this, from Lemma 10, and from Fubini's theorem it follows that  $\mathcal{H}^\eta(K \cap V_z^t) = 0$  for  $\nu$ -a.e.  $(z, t) \in X$ , which completes the proof of Theorem 15.  $\square$

*Proof of Theorem 6:* Let  $G$  be the set of all 1-dimensional linear subspaces of  $\mathbb{R}^2$ , and set

$$E = \{(z, V) \in K \times G : \mathcal{H}^{d_a+d_b-1}(K \cap V_z) = 0\}.$$

For each  $-\infty \leq t_1 < t_2 \leq \infty$  set

$$G_{t_1, t_2} = \{V \in G : V = (t, -1) \cdot \mathbb{R} \text{ with } t \in (t_1, t_2)\}.$$

Given  $k \in \mathbb{Z}$  we can apply theorem 15 with  $\tau = a^k$ , in order to get that  $(z, V) \in E$  for  $\mu \times \xi_G$ -a.e.  $(z, V) \in K \times G_{a^{k+1}, a^k}$ . By doing this for each  $k \in \mathbb{Z}$  we get that  $(z, V) \in E$  for  $\mu \times \xi_G$ -a.e.  $(z, V) \in K \times G_{0, \infty}$ . Now Theorem 6 follows from the symmetry of  $K$  with respect to the map that takes  $(x, y) \in K$  to  $(1-x, y)$ .  $\square$

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